ON DECOMPOSITION NUMBERS AND ALVIS-CURTIS DUALITY

BERND ACKERMANN AND SIBYLLE SCHROLL

Abstract. We show that for general linear groups $\mathrm{GL}_n(q)$ as well as for q-Schur algebras the knowledge of the modular Alvis-Curtis duality over fields of characteristic ℓ, $\ell \nmid q$, is equivalent to the knowledge of the decomposition numbers.

1. Introduction

In the beginning of the nineties, Broué's abelian defect group conjecture [2] related the homological and the character theoretic aspects of representation theory. It follows from the conjecture that certain character correspondences should in fact be consequences of derived equivalences. A related phenomenon is the Alvis-Curtis duality. Originally it was defined as a character duality of finite groups of Lie type, which in particular sends an irreducible character onto another irreducible character up to sign. Broué has shown [2] that it is what he calls a perfect isometry, which is a character correspondence with signs and some arithmetic properties. In '99 Cabanes and Rickard [6] then showed that the Alvis-Curtis duality is in fact induced by a derived equivalence obtained by tensoring with a certain complex.

On the other hand, questions on perfect isometries naturally lead to questions on decomposition numbers. These numbers are virtually unknown for almost all groups. In particular, the question of how to calculate the decomposition matrices of the general linear group has not yet been satisfactorily answered. James gave an algorithm for calculating the decomposition matrices of $\mathrm{GL}_n(q)$, for $n \leq 10$, in non-describing characteristic [16]. But for larger n there is no algorithm for the calculation of the decomposition matrix of $\mathrm{GL}_n(q)$. Similary, except for small cases [11, 19, 20, 17, 26], the decomposition matrices of the q-Schur and the Hecke algebra are not known. In this paper, we show that the complete knowledge of the Alvis-Curtis duality answers these questions for the general linear group and the q-Schur algebra.

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More precisely, we construct a cochain complex for the q-Schur algebra. We then relate it to the complex inducing the Alvis-Curtis duality [6] as well as to a complex for the Hecke algebra [18]. We show that this complex for the q-Schur algebra induces a self-derived equivalence and thus gives rise to a duality operation in the Grothendieck group. We analyse how the different duality operators arising in this context act on the different simple modules in the modular case. In fact, we show that calculating this action for the general linear group and for the q-Schur algebra is equivalent to calculating the decomposition matrices of these algebras.

In the following we describe the layout of the paper. Section 2 contains notation and background results needed later on. In section 3, Theorem 3.2, we show the main result for general linear groups. A link with the Mullineux map and the Hecke algebra is established in Theorem 4.1 in section 4. Section 5 contains the construction of the complex for the q-Schur algebra. In Theorem 5.1 we show that it induces a derived equivalence and we calculate the induced action on the simple modules in the Grothendieck group. Finally in Theorem 6.1 in section 6 we give a summary of the results on the different decomposition matrices.

2. Preliminaries

Let $G = \mathrm{GL}_n(q)$ and $(K, \mathcal{O}, k)$ an ℓ-modular system with ℓ coprime to q. Let e be the smallest integer i such that modulo ℓ

$$1 + q + \dots q^{i-1} = 0$$

Throughout we fix the following notation. Denote by T the maximal torus of invertible diagonal matrices in G, by U the group of upper unitriangular matrices and set $B = UT$. Let W be the Weyl group of G. Then $W \cong \mathfrak{S}_n$ and we identify $\mathfrak{S}_n$ with the subgroup of permutation matrices of G. Denote by S the generating set of $\mathfrak{S}_n$ corresponding to the set of basic transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$. For any composition λ of n denote by $\mathfrak{S}_\lambda$ the associated Young subgroup of $\mathfrak{S}_n$ and by P_λ the standard parabolic subgroup of G generated by B and $\mathfrak{S}_\lambda$. Denote by U_λ the unipotent radical of P_λ and by L_λ the standard Levi complement of U_λ in P_λ. For any subgroup H of G let e_H be the idempotent of KG defined by $e_H = \frac{1}{|H|} \sum\limits_{x \in H} x$. Note that if H is an ℓ'-subgroup then $e_H \in RG$ for R any of $\{K, \mathcal{O}, k\}$. Denote by $\mathcal{E}(G, 1)$ the unipotent character series, that is the irreducible constituents of $\mathrm{Ind}_B^G(K)$. Define the central idempotent $f = \sum_\chi e(\chi)$ of KG where χ runs over the set of unipotent characters of G and

where $e(\chi) = \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g^{-1})g$. Then f is the block idempotent of the sum of all unipotent ℓ-blocks. In a similar way define f_λ as the sum of the $e(\psi)$ where ψ runs through the unipotent characters of L_λ. Let G_{ss} be the set of semisimple elements of G and G_{ss}^{reg} the set of ℓ-regular semisimple elements of G. By $(g)\in_G G$ we denote the conjugacy class of g in G.

For $R\in\{K,\mathcal{O},k\}$ and any composition λ of n denote by $R_{L_\lambda}^G$ the Harish-Chandra induction functor that associates to a RL_λ-module N the RG-module $RGe_{U_\lambda}\otimes_{RL_\lambda} N$. Its left and right adjoint called Harish-Chandra restriction and denoted by ${}^*\mathrm{R}_{L_\lambda}^G$ associates to a RG-module M the RL_λ-module $e_{U_\lambda}M$.

2.1. **Combinatorics.** Denote by $\bar\Lambda(k,n)$ the set of compositions of n with exactly k non-zero parts and denote by $\bar\Lambda^+(k,n)$ the subset containing compositions $\lambda=(\lambda_1,\ldots,\lambda_k)$ such that $\lambda_1\geq\ldots\geq\lambda_k$. Let $\Lambda(k,n)=\bigoplus_{i\leq k}\bar\Lambda(i,n)$ and $\Lambda^+(k,n)=\bigoplus_{i\leq k}\bar\Lambda^+(i,n)$. Write $\Lambda(n)$ for $\Lambda(n,n)$ and similarly $\Lambda^+(n)$ for $\Lambda^+(n,n)$. For $\nu=(\nu_1,\ldots,\nu_a)\in\Lambda(n)$ define $\Lambda(\nu)$ as the subset of all compositions $\gamma=(\gamma_1,\ldots,\gamma_h)\in\Lambda(n)$ such that $(\gamma_1,\ldots,\gamma_{h_1})\in\Lambda(\nu_1)$, $(\gamma_{h_1+1},\ldots,\gamma_{h_1+h_2})\in\Lambda(\nu_2)$ and so on until $(\gamma_{h_1+\ldots+h_{a-1}+1},\ldots,\gamma_h)\in\Lambda(\nu_a)$. Write $|\lambda|$ for the number of non-zero parts of a composition λ. An e-regular partition of n is a partition $\lambda=(\lambda_1,\ldots,\lambda_a)$ such that if $\lambda_{i+1}=\lambda_{i+2}=\ldots=\lambda_{i+h}$ then $h<e$.

There exists a partial order on the set of compositions of n induced by the partial order on the generating set S of $\mathfrak{S}_n$ given by inclusion of subsets. More precisely, the subsets of S are in bijection with compositions of n : Let $\lambda=(\lambda_1,\ldots,\lambda_h)$ be a composition of n and set

$$\lambda_i^+=\lambda_1+\lambda_2+\ldots+\lambda_i$$

for $1\leq i\leq h$. Then λ corresponds to the subset I_λ of S given by $I_\lambda=\{s_1,\ldots,s_n\}\setminus\{\lambda_i^+|1\leq i\leq h\}$. Note that $|I_\lambda|=(n-1)-(h-1)=n-h$. An easy calculation then shows

Lemma 2.1. *Suppose I_λ and I_μ are subsets of S corresponding to $\lambda\in\bar\Lambda(h,n)$ and $\mu\in\bar\Lambda(k,n)$. Then*

(i) $|I_\lambda|\leq|I_\mu|$ if and only if $h\geq k$

(ii) $I_\lambda\subset I_\mu$ if and only if for all μ_i^+ there exists λ_j^+ such that $\mu_i^+=\lambda_j^+$.

For compositions satisfying (ii) we write $\lambda\preceq\mu$. Note that this is a partial order and that $\lambda\preceq\mu$ implies $\lambda\trianglelefteq\mu$ in the dominance order.

2.2. **Simple modules of the general linear group.** Let s be a semisimple element of G. Then the rational canonical form of s is some block diagonal matrix where the diagonal blocks are given as companion matrices (σ_i) of the elementary divisors s_i of s which are the irreducible factors of the minimal polynomial of s over $\mathrm{GF}(q)$ since s is semisimple. Assume that (σ_i) appears precisely k_i times on the diagonal and let d_i be the degree of σ_i, then (σ_i) is a $d_i \times d_i$-matrix and $d_1k_1 + \cdots + d_rk_r = n$. The centralizer $C_G(s)$ of s in G is then isomorphic to

$$C_G(s) \cong \mathrm{GL}_{k_1}(q^{d_1}) \times \ldots \times \mathrm{GL}_{k_r}(q^{d_r}).$$

Details and proofs can be found in [FS]. With these notations define $\varepsilon_s = (-1)^{n+k_1+\ldots+k_r}$.

Let $\lambda^{(i)}$ be a partition of k_i. Following James [15] we have an irreducible $K\mathrm{GL}_{d_ik_i}(q)$-module $S(s_i, \lambda^{(i)})$ called Specht module and each irreducible KG-module is of the form

$$S(s, \tilde{\lambda}) = R_L^G \left(S(s_1, \lambda^{(1)}) \otimes \cdots \otimes S(s_r, \lambda^{(r)}) \right)$$

where $\tilde{\lambda}$ is the multipartition $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ of $(k_1, \ldots, k_r)$ and L is the Levi subgroup $\mathrm{GL}_{d_1k_1}(q) \times \cdots \times \mathrm{GL}_{d_rk_r}(q)$. Note that the ordinary irreducible character of $S(s, \tilde{\lambda})$ corresponds exactly to the character $\chi_{s,\tilde{\lambda}}$ in the parametrization given by Green [12] and the set

$$\{S(s, \tilde{\lambda}) | (s) = (s_i^{k_i}) \in_G G_{ss}, \tilde{\lambda} \vdash (k_1, \ldots, k_r)\}$$

gives a complete set of irreducible KG-modules [15].

In [15] James constructs a certain OG-lattice $S_O(s, \tilde{\lambda})$ in $S_K(s, \tilde{\lambda})$. It is shown there that the ℓ-modular reduction $S_F(s, \tilde{\lambda})$ of this lattice no longer needs to be irreducible. However, if s is ℓ-regular it has an irreducible head $L(s, \tilde{\lambda})$ and we have

$$L(s, \tilde{\lambda}) = R_L^G \left(L(s_1, \lambda^{(1)}) \otimes \cdots \otimes L(s_r, \lambda^{(r)}) \right)$$

where each $L(s_i, \lambda^{(i)})$ is the irreducible head of $S(s_i, \lambda^{(i)})$. Moreover, the set

$$\{L(s, \tilde{\lambda}) | (s) = (s_i^{k_i}) \in_G G_{ss}^{reg}, \tilde{\lambda} \vdash (k_1, \ldots, k_r)\}$$

is a complete set of irreducible kG-modules.

2.3. **Hecke algebras.** Let $R \in \{K, \mathcal{O}, k\}$ and denote by $\mathcal{H}_R$ or by $\mathcal{H}$ the Iwahori Hecke algebra $\mathcal{H}_{R,q}(\mathfrak{S}_n)$. This is defined as the free R-module with basis $\{T_w | w \in S_n\}$ and with multiplication given by

$$T_wT_s = \begin{cases} T_{ws} & \text{if } \ \ell(ws) > \ell(w) \\ qT_w + (q-1)T_{ws} & \text{if } \ \ell(ws) < \ell(w) \end{cases}$$

for $w \in \mathfrak{S}_n$ and $s \in S$. For every composition λ of n we denote by $\mathcal{H}_\lambda$ the parabolic subalgebra of $\mathcal{H}$ isomorphic to the Iwahori Hecke algebra $\mathcal{H}_{R,q}(\mathfrak{S}_\lambda)$.

For every parabolic subalgebra $\mathcal{H}_\lambda$ there is an induction functor

$$\begin{array}{rcccc} \mathrm{Ind}_{\mathcal{H}_\lambda}^{\mathcal{H}} & : & \mathcal{H}_\lambda - \mathrm{mod} & \to & \mathcal{H} - \mathrm{mod} \\ & & N & \mapsto & \mathcal{H} \otimes_{\mathcal{H}_\lambda} N \end{array}$$

This is an exact functor and its left and right adjoint is given by

$$\begin{array}{rcccc} \mathrm{Res}_{\mathcal{H}_\lambda}^{\mathcal{H}} & : & \mathcal{H} - \mathrm{mod} & \to & \mathcal{H}_\lambda - \mathrm{mod} \\ & & M & \mapsto & \mathcal{H}_\lambda \otimes_{\mathcal{H}} N \end{array}$$

A full set of irreducible $\mathcal{H}_K$-modules is parametrised as follows

$$\{S^\lambda | \lambda \in \Lambda^+(n)\}.$$

As $\mathcal{H}_k$-modules these need no longer be irreducible; however, if the partition λ is e-regular, then S^λ has an irreducible head L^λ and the set

$$\{L^\lambda | \lambda \in \Lambda^+(n),\ \lambda \text{ is } e\text{-regular}\}$$

is a complete set of irreducible $\mathcal{H}_k$-modules [8].

In [13] an involution $\alpha : \mathcal{H} \to \mathcal{H}$ is defined by $T_w \mapsto (-q)^{\ell(w)}(T_{w^{-1}})^{-1}$ for $w \in \mathfrak{S}_n$. For an $\mathcal{H}$-module V define another $\mathcal{H}$-module ${}_\alpha V$ by setting $h.v = \alpha(h)v$ for all $h \in \mathcal{H}$ and $v \in {}_\alpha V$. Then if $q = 1$, ${}_\alpha V = sign \otimes V$, where $sign$ is the sign representation of $\mathfrak{S}_n$. Furthermore it is shown in [3] that for any e-regular partition λ of n and any simple $\mathcal{H}$-module L^λ we have ${}_\alpha(L^\lambda) = L^{m(\lambda)}$ where $m(\lambda)$ is the image of λ by the Mullineux map m (for the definition of m see [21]).

2.4. **q-Schur algebras.** For each composition λ of n there is a permutation module M^λ of $\mathcal{H} = \mathcal{H}_R$ for $R \in \{K, \mathcal{O}, k\}$ given by $M^\lambda = \mathcal{H}x_\lambda$ where $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w$. The q-Schur algebra $\mathcal{S}_R(k,n)$ is the endomorphism algebra $\mathrm{End}_{\mathcal{H}}(\bigoplus_{\lambda \in \Lambda(k,n)} M^\lambda)$. We are mostly interested in the q-Schur algebra $\mathcal{S}_R(n,n)$ and certain of its subalgebras. Note however, that for $k \geq n$ there is a Morita equivalence between $\mathcal{S}_R(k,n)$ and $\mathcal{S}_R(n)$. We write $\mathcal{S}_R(n)$ or $\mathcal{S}(n)$ for $\mathcal{S}_R(n,n)$. The algebra $\mathcal{S}(n)$ is free over R and has basis

$$\{\Phi^u_{\lambda,\mu} | \lambda, \mu \in \Lambda(n), u \in \mathcal{D}_{\lambda,\mu}\}$$

where $\mathcal{D}_{\lambda,\mu}$ is the set of distinguished double coset representatives of $\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\mu$ which are the unique elements of shortest length in their double coset [9]. The elements $\Phi^1_{\lambda,\lambda}$ for $\lambda \in \Lambda(n)$ are orthogonal idempotents in $\mathcal{S}(n)$. If $\lambda = (1^n)$ then multiplication by the idempotent $e = \Phi^1_{\lambda,\lambda}$ is called the Schur functor and $e\mathcal{S}(n)e \cong \mathcal{H}$. Furthermore,

for each composition ν, we define a generalization of the Schur functor. That is, we define an idempotent of $\mathcal{S}(n)$

$$e_\nu = \sum_{\lambda\in\Lambda(\nu)} \Phi^1_{\lambda,\lambda}.$$

The subalgebra $\mathcal{S}(\nu) = \mathcal{S}_R(\nu_1)\otimes\ldots\otimes\mathcal{S}_R(\nu_h)$ naturally embeds into $e_\nu\mathcal{S}(n)e_\nu$ (see for example [4], section 4.2 for details) and we can define the following functors

$$\begin{aligned} \mathcal{S}(n)e_\nu\otimes_{\mathcal{S}(\nu)} - &: \mathcal{S}(\nu)-\mathrm{mod} \rightarrow \mathcal{S}(n)-\mathrm{mod} \\ e_\nu.- &: \mathcal{S}(n)-\mathrm{mod} \rightarrow \mathcal{S}(\nu)-\mathrm{mod}. \end{aligned}$$

The q-Schur algebra $\mathcal{S}_k(n)$ is quasi-hereditary, the system of labels of the standard modules being partitions of n ordered by the dominance ordering. The general theory of quasi-hereditary algebras produces a parametrisation of the simple $\mathcal{S}_k(n)$-modules $L(\lambda)$, $\lambda\in\Lambda^+(n)$. Furthermore, the set $\Lambda^+(n)$ also parametrizes the so-called standard modules $\Delta(\lambda)$ and the costandard modules $\nabla(\lambda)$.

By Takeuchi [25] and Brundan, Dipper, Kleshchev [4] the q-Schur algebra $\mathcal{S}_R(n)$ is Morita equivalent to the quotient algebra $C_R(n) = RG/I$ where I is the annihilator of the permutation module $R(G/B)$ on the cosets G/B of B in G. In fact the Morita equivalence is given by a RG-$\mathcal{S}_R(n)$-bimodule Q and its $C_R(n)$-linear dual (see [4, 3.4g] for the precise definition). Furthermore, $C_{\mathcal{O}}(n) = \mathcal{O}G/I$ is isomorphic to $\mathcal{O}Gf$ (see [24] Thm 1 for a proof) and $fQ = Q$ thus Q is also an $\mathcal{O}Gf$-module where f is again the central idempotent of the sum of unipotent characters defined in the beginning of section 2. Furthermore

$$\begin{aligned} Q\otimes_{\mathcal{S}(n)} - &: \mathcal{S}(n)-\mathrm{mod} \rightarrow \mathcal{O}Gf-\mathrm{mod} \\ Q'\otimes_{\mathcal{O}Gf} - &: \mathcal{O}Gf-\mathrm{mod} \rightarrow \mathcal{S}(n)-\mathrm{mod} \end{aligned}$$

are mutually inverse functors, where Q' denotes the $\mathcal{O}Gf$-linear dual of Q.

2.5. **Alvis-Curtis duality.** Originally Alvis-Curtis duality was introduced as a duality operator on the ordinary character group of a finite group of Lie type. Deligne and Lusztig defined it as the Lefschetz character of a certain complex of the module category in characteristic 0. Cabanes and Rickard [6] formalized this definition by using coefficient systems and extended it to representations over any ring containing

p^{-1}, where $q = p^\alpha$. Although their definition holds for all finite groups of Lie type we will present it here only for $G = \mathrm{GL}_n(q)$.

Let X_G be the complex associated to the coefficient system on the simplicial complex given by the set of compositions of n with the partial order $\preceq$. It sends a composition λ to the RG-RG-bimodule $RGe_{U_\lambda} \otimes_{RL_\lambda} e_{U_\lambda} RG$ and the inclusion $\lambda \prec \nu$ to the map

$$\begin{array}{rccc} \alpha_{\lambda,\nu}: & RGe_{U_\lambda} \otimes_{RL_\lambda} e_{U_\lambda} RG & \rightarrow & RGe_{U_\nu} \otimes_{RL_\nu} e_{U_\nu} RG \\ & x \otimes y & \mapsto & x \otimes y. \end{array}$$

The associated cochain complex X_G, shifted and renumbered, such that it is concentrated in degrees 0 to $n-1$ has degree ith term

$$X_G^i = \bigoplus_{\lambda \in \bar{\Lambda}(n-i,n)} RGe_{U_\lambda} \otimes_{RL_\lambda} e_{U_\lambda} RG$$

for $0 \leq i \leq n-1$. Its differential is given by

$$d^i = \sum_{\lambda \in \bar{\Lambda}(n-i,n)} \sum_{\substack{\nu \in \bar{\Lambda}(n-i-1,n) \\ \lambda \prec \nu}} (-1)^{\lambda/\nu} \alpha_{\lambda,\nu}$$

where λ/ν is the unique partial sum λ_j^+ of λ such that there is no partial sum ν_k^+ of ν such that $\lambda_j^+ = \nu_k^+$.

In [6] Cabanes and Rickard showed that tensoring with X_G induces a derived self-equivalence of the module category of G. Therefore we obtain an induced map in the Grothendieck group of G, that is a bijection in characteristic 0,

$$D_G(-) = \sum_{\lambda \in \Lambda(n)} (-1)^{|\lambda|} RGe_{U_\lambda} \otimes_{RL_\lambda} e_{U_\lambda}(-).$$

For any standard Levi subgroup $L_\lambda = \mathrm{GL}_{\lambda_1}(q) \times \ldots \times \mathrm{GL}_{\lambda_a}(q)$ we define

$$X_{L_\lambda} = X_{\mathrm{GL}_{\lambda_1}(q)} \otimes \ldots \otimes X_{\mathrm{GL}_{\lambda_a}(q)}.$$

In fact, it is easy to see that X_{L_λ} coincides with the complex defined for L_λ when it is itself considered as a finite group of Lie type.

It then follows from [6] that in the Grothendieck group Alvis-Curtis duality commutes with Harish-Chandra induction and restriction :

Lemma 2.2. [6, 6.1] *Let L be a standard Levi subgroup of G. Then*

$$R_L^G \circ D_L[N] = D_G \circ R_L^G[N]$$

$${}^*\mathrm{R}_L^G \circ D_G[M] = D_L \circ {}^*\mathrm{R}_L^G[M]$$

for $[N]$ *in* $K_0(RL)$ *and* $[M]$ *in* $K_0(RG)$ *with* $R \in \{K, \mathcal{O}, k\}$*.*

3. Modular Alvis-Curtis duality

In the case of the general linear group G the Alvis-Curtis dual of an irreducible KG-module is well-understood (see below). In the modular case however, little is known. For example it is open what the Alvis-Curtis dual of a simple kG-module is. Indeed it turns out that this question can be transformed into a question on decomposition matrices of general linear groups. More precisely we will show that determining the Alvis-Curtis duality on simple modules for general linear groups in non-describing characteristic is equivalent to determining the decomposition matrix of G.

For the convenience of the reader we recall the result on the simple KG-modules (see for example [23]).

Proposition 3.1. *Let* $S(s, \tilde{\lambda})$ *be an irreducible* KG*-module. Then in the Grothendieck group* $K_0(KG)$

$$D_G([S(s, \tilde{\lambda})]) = \varepsilon_s[S(s, \tilde{\lambda}')]$$

where $\tilde{\lambda}' = (\lambda^{(1)'}, \ldots, \lambda^{(r)'})$ *denotes the conjugate multipartition of* $\tilde{\lambda}$ *and* ε_s *is defined as in section 2.2.*

Assume that the set of partitions $\Lambda^+(n)$ is ordered by a linear order $\leq$ refining the dominance order, for example the lexicographic order. Denote by $\mathcal{Z}_G$ the decomposition matrix of G and let $\mathcal{Z}_u = (z_{\lambda,\nu})$ be the upper (quadratic) part of $\mathcal{Z}_G$ restricted to the unipotent block. So we have that $\mathcal{Z}_u = (z_{\lambda,\nu})$ with $z_{\lambda,\nu} = [S(1, \lambda) : L(1, \nu)]$ for partitions λ, ν of n.

Denote by $A_G = (a_{\lambda,\nu})$ the matrix of the Alvis-Curtis duality on the unipotent kG-modules, that is $a_{\lambda,\nu}$ is the coefficient of $[L(1, \nu)]$ in $D_G[L(1, \lambda)]$. Finally let P be the permutation matrix given by the permutation on $\Lambda^+(n)$ that sends λ to λ'.

Then the decomposition matrix of G determines the Alvis-Curtis duality and the knowledge of the Alvis-Curtis duality for smaller general linear groups determines the decomposition matrix of G. More precisely, we have

Theorem 3.2. *Let $G = \mathrm{GL}_n(q)$. Then with the notations above*

(i) A_G is determined by the decomposition matrix of G and A_G determines the unipotent part $\mathcal{Z}_u$ of the decomposition matrix of G. Explicitly, $A_G = \mathcal{Z}_u^{-1} \cdot P \cdot \mathcal{Z}_u$.

(ii) Alvis-Curtis duality on $K_0(k\mathrm{GL}_k(q^d))$ for all d, k such that $dk \leq n$ determines the whole of the decomposition matrix $\mathcal{Z}_G$ of G.

(iii) The decomposition matrix $\mathcal{Z}_G$ of G determines the Alvis-Curtis duality on all irreducible kG-modules.

Proof. (i) Proposition 3.1 holds also for modular Specht modules and thus $D_G([S(1,\lambda)]) = [S(1,\lambda')]$ in $K_0(kG)$. This implies $\mathcal{Z}_u \cdot A_G = P \cdot \mathcal{Z}_u$. Since $\mathcal{Z}_u$ is lower unitriangular (see [14, 16.3]) it is invertible, so $A_G = \mathcal{Z}_u^{-1} P \mathcal{Z}_u$ and A_G is determined by $\mathcal{Z}_G$.

On the other hand, since A_G is invertible we can use the Bruhat decomposition (see [5, 2.5.13]) to write $A_G = U_1 \cdot T \cdot R \cdot U_2$ in a unique way, where U_1, U_2 are lower unitriangular, T is a diagonal matrix and R is a permutation matrix such that $R \cdot U_2 \cdot R^{-1}$ is upper uni-triangular. We already know that $\mathcal{Z}_u$ and therefore $\mathcal{Z}_u^{-1}$ are lower uni-triangular. Hence $T = 1$, $R = P$ and to complete the proof that $U_2 = \mathcal{Z}_u$ it remains to show that $P\mathcal{Z}_u P^{-1}$ is upper unitriangular. By [14] Cor. 16.3 we know that $(P\mathcal{Z}_u P^{-1})_{\lambda\mu} = d_{\lambda'\mu'} = 0$ unless $\lambda' \trianglerighteq \mu'$. But this is equivalent to $\lambda \trianglelefteq \mu$ and therefore $P\mathcal{Z}_u P^{-1}$ is upper unitriangular. So $\mathcal{Z}_u$ is determined by A_G as the unique lower unitriangular matrix in the Bruhat decomposition of A_G.

(ii) By the work of Dipper and James [9] we know that the upper (unitriangular) parts of the decomposition matrices of the unipotent blocks of the groups $\mathrm{GL}_k(q^d)$ for $kd \leq n$ determine the decomposition matrix of G. So *(ii)* is an easy consequence of part *(i)*.

(iii) The decomposition matrix $\mathcal{Z}_G$ has an upper (quadratic) part that is lower unitriangular (see [7]). Therefore every irreducible kG-module in $K_0(kG)$ can be written as integral linear combination of Specht modules. By Proposition 3.1 we know the duality on Specht modules. Therefore as in (i) the decomposion matrix determines the duality operator. □

We would like to point out that the converse to Theorem 3.2(iii) is not true in general since the upper (quadratic) part of the decomposition matrix does not determine the whole of the decomposition matrix. To calculate the whole decomposition matrix of $\mathrm{GL}_n(q)$ one needs to know all the quadratic parts of all decomposition matrices for all groups $\mathrm{GL}_k(q^d)$ where $dk \leq n$. In terms of Alvis-Curtis duality this equals the

knowledge of the duality operator on all these groups as in Theorem 3.2(ii).

4. Modular Alvis-Curtis duality on Hecke algebras

Let $\mathcal{H} = \mathcal{H}_k$. Recall from [18] the complex $X_{\mathcal{H}}$ of $\mathcal{H}$-$\mathcal{H}$-bimodules defined by

$$X^i_{\mathcal{H}} = \bigoplus_{\lambda \in \bar{\Lambda}(n-i,n)} \mathcal{H} \otimes_{\mathcal{H}_\lambda} \mathcal{H}$$

for $0 \leq i \leq n-1$ and with differential

$$d^i = \sum_{\lambda \in \bar{\Lambda}(n-i,n)} \sum_{\substack{\nu \in \bar{\Lambda}(n-i-1,n) \\ \lambda \prec \nu}} (-1)^{\lambda/\nu} \vartheta_{\lambda,\nu}$$

where $\vartheta_{\lambda,\nu} : \mathcal{H} \otimes_{\mathcal{H}_\lambda} \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{H}_\nu} \mathcal{H}$ is the canonical surjection.

Note that this complex can be constructed as the chain complex associated to a coefficient system on the same simplicial complex as for X_G.

Furthermore, it is shown in [18] that tensoring with $\mathcal{H}$ induces a derived self-equivalence of the module category of $\mathcal{H}$, that the cohomology of $\mathcal{H}$ is concentrated in degree 0 and that it is isomorphic to ${}_\alpha\mathcal{H}$ as an $\mathcal{H}$-$\mathcal{H}$-bimodule.

Therefore $X_{\mathcal{H}}$ induces a duality operator $D_{\mathcal{H}}$ in the Grothendieck group $K_0(\mathcal{H})$ given by

$$(1) \quad D_{\mathcal{H}}([V]) = \sum_{\lambda \in \Lambda(n)} (-1)^{|\lambda|} \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_\lambda} \circ \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}_\lambda}[V] = [{}_\alpha\mathcal{H} \otimes_{\mathcal{H}} V] = [{}_\alpha V]$$

for $[V] \in K_0(\mathcal{H})$.

In [24] it was shown that the complex $X_{\mathcal{H}}$ is isomorphic to a quotient of the complex X_G, namely to $e_U X_G f e_U$. This relation on the level of complexes is reflected on the level of the Grothendieck groups in the following form. Denote by $A_{\mathcal{H}} = (h_{\lambda,\nu})$ the matrix whose entries are given by the coefficients of $[L^\nu]$ in a decomposition of $D_{\mathcal{H}}([L^\lambda])$ for λ, ν e-regular partitions of n.

Theorem 4.1. *The matrix $A_G = (a_{\lambda,\nu})$ giving the Alvis-Curtis duality completely determines the matrix $A_{\mathcal{H}}$. More precisely, for λ, ν e-regular partitions of n we have*

$$a_{\lambda,\nu} = h_{\lambda,\nu} = \begin{cases} 1 & \text{if } \nu = m(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

For the proof of theorem 4.1 we need the following lemma, which is a direct consequence of [4], Corollary 3.2(g) that:

Lemma 4.2. *Let M denote the permutation module $M = \mathrm{R}_T^G(k)$. Then the following diagram commutes:*

$$\begin{tikzcd} \mathcal{H}-mod \arrow[r, "M\otimes_{\mathcal{H}}-"] \arrow[d, "\mathrm{Ind}_{\mathcal{H}_\lambda}^{\mathcal{H}}\circ\mathrm{Res}_{\mathcal{H}_\lambda}^{\mathcal{H}}"'] & kG-mod \arrow[d, "\mathrm{R}_{L_\lambda}^G\circ{}^*\mathrm{R}_{L_\lambda}^G"] \\ \mathcal{H}-mod \arrow[r, "M\otimes_{\mathcal{H}}-"] & kG-mod \end{tikzcd}$$

Proof. This is Cor. 3.2(g) of [4]. □

Proof of Theorem 4.1. Let β denote the functor $M \otimes_{\mathcal{H}} \_$. Since β is exact Lemma 4.2 gives us $D_G \circ \beta = \beta \circ D_{\mathcal{H}}$ on the level of Grothendieck groups. And since $\beta(L^\lambda) \cong L(1,\lambda)$ we have

$$\sum_\mu a_{\lambda,\mu}[L(1,\mu)] = D_G \circ \beta([L^\lambda])$$
$$= \beta \circ D_{\mathcal{H}}([L^\lambda]) = \beta(\sum_\mu h_{\lambda,\mu}[L^\mu] = \sum_\mu h_{\lambda,\mu}[L(1,\mu)]$$

which proves the first equality. The second equality follows directly from formula (1).

□

5. Modular Alvis-Curtis duality on q-Schur algebras

We define a duality operator on the Grothendieck group of $\mathcal{S}(n)$-modules induced by a derived self-equivalence of the module category given by tensoring with a complex. This duality operator is closely linked to the Alvis-Curtis duality operator on the level of the Grothendieck group as well as on the level of the derived category. Furthermore, we show how this complex relates to the complex $X_{\mathcal{H}}$ for the Hecke algebra defined in the previous section.

Let $\mathcal{S}(n) = \mathcal{S}_R(n)$ be the q-Schur algebra defined over $R \in \{\mathcal{O}, k\}$. Define the cochain complex $X_{\mathcal{S}}$ of $\mathcal{S}(n)$-$\mathcal{S}(n)$-bimodules as the cochain

complex associated to the coefficient system on the simplicial complex of compositions of n with partial order $\preceq$. A composition λ of n is sent to the $\mathcal{S}(n)$-$\mathcal{S}(n)$-bimodule $\mathcal{S}(n)e_\lambda \otimes_{S(\lambda)} e_\lambda \mathcal{S}(n)$ and the inclusion $\lambda \prec \nu$ is sent to the map

$$\begin{array}{cccc} \gamma_{\lambda,\nu} : & \mathcal{S}(n)e_\lambda \otimes_{S(\lambda)} e_\lambda \mathcal{S}(n) & \to & \mathcal{S}(n)e_\nu \otimes_{S(\nu)} e_\nu \mathcal{S}(n) \\ & x \otimes y & \mapsto & x \otimes y. \end{array}$$

Note that for λ, ν as above, $e_\nu e_\lambda = e_\lambda e_\nu = e_\nu$ thus the maps are well-defined. Hence after shifting and renumbering the complex $X_{\mathcal{S}}$ is concentrated in degrees 0 to $n-1$ and for $0 \leq i \leq n-1$

$$X^i_{\mathcal{S}} = \bigoplus_{\lambda \in \bar{\Lambda}(n-i,n)} \mathcal{S}(n)e_\lambda \otimes_{\mathcal{S}(\lambda)} e_\lambda \mathcal{S}(n).$$

The differential is given by

$$d^i = \sum_{\lambda \in \bar{\Lambda}(n-i,n)} \sum_{\substack{\nu \in \bar{\Lambda}(n-i-1,n) \\ \lambda \prec \nu}} (-1)^{\lambda/\nu} \gamma_{\lambda,\nu}.$$

Then $X_{\mathcal{S}}$ induces a derived self-equivalence and a duality operation in the Grothendieck group of the module category of $\mathcal{S}(n)$.

Theorem 5.1. *The functor $X_{\mathcal{S}} \otimes_{\mathcal{S}(n)} - : \mathcal{D}^\flat(\mathcal{S}(n)) \to \mathcal{D}^\flat(\mathcal{S}(n))$ is an equivalence of categories inducing a duality operator $D_{\mathcal{S}}$ of the Grothendieck group $K_0(\mathcal{S}_k(n))$ with the following properties*

(i) $D_{\mathcal{S}}(\Delta(\lambda)) = \Delta(\lambda')$ for all standard modules $\Delta(\lambda)$.

(ii) $D_{\mathcal{S}}(\nabla(\lambda)) = \nabla(\lambda')$ for all costandard modules $\nabla(\lambda)$.

(iii) $D_{\mathcal{S}}(L(\lambda)) = \sum_{\mu \in \Lambda^+(n)} a_{\lambda'\mu'} L(\mu)$, where $A_G = (a_{\lambda\mu})$ is the matrix given by the Alvis-Curtis duality.

Theorem 5.2. *There is a split monomorphism of complexes of $\mathcal{H}_R$-$\mathcal{H}_R$-bimodules from $X_{\mathcal{H}}$ into $eX_{\mathcal{S}}e$.*

Proof. Let $\mathcal{H} = \mathcal{H}_R$ and recall that $e\mathcal{S}(n)e \cong \mathcal{H}$. By [10, 4.6(5)] the idempotent e also satisfies $e\mathcal{S}(\lambda)e \cong \mathcal{H}_\lambda$ for all compositions λ of n. Therefore the terms of the complex $X_{\mathcal{H}}$ are isomorphic to direct sums of terms of the form $e\mathcal{S}(n)e \otimes_{e\mathcal{S}(\lambda)e} e\mathcal{S}(n)e$. The map defined by

$$\begin{array}{ccc} e\mathcal{S}(n)e_\lambda \otimes_{\mathcal{S}(\lambda)} e_\lambda \mathcal{S}(n)e & \to & e\mathcal{S}(n)e \otimes_{e\mathcal{S}(\lambda)e} e\mathcal{S}(n)e \\ x \otimes y & \mapsto & xe \otimes ey \end{array}$$

induces then a morphism of complexes. It is a splitting map for the injection inducing a monomorphism of complexes

$$\begin{array}{ccc} e\mathcal{S}(n)e \otimes_{e\mathcal{S}(\lambda)e} e\mathcal{S}(n)e & \to & e\mathcal{S}(n)e_\lambda \otimes_{\mathcal{S}(\lambda)} e_\lambda\mathcal{S}(n)e \\ x \otimes y & \mapsto & x \otimes y. \end{array}$$

□

Before proving theorem 5.1 let us give some preliminary results.

By [4, 4.2c] the functor $\mathcal{S}(n)e_\lambda \otimes_{\mathcal{S}(\lambda)} -$ corresponds to Harish-Chandra induction in the general linear group. More precisely, if M is an $\mathcal{S}(\lambda)$-$\mathcal{S}(n)$-bimodule then there is an ismorphism of $C(n)$-$\mathcal{S}(n)$-bimodules

$$R^G_{L_\lambda}(Q_\lambda \otimes_{\mathcal{S}(\lambda)} M) \cong Q \otimes_{\mathcal{S}(n)} \mathcal{S}(n)e_\lambda \otimes_{\mathcal{S}(\lambda)} M \cong Qe_\lambda \otimes_{\mathcal{S}(\lambda)} M.$$

Here Q_λ is the $\mathcal{S}(\lambda)$-RL_λ bimodule inducing a Morita equivalence between $\mathcal{S}(\lambda)$ and $C(\lambda)$. That is the following diagram commutes

$$\begin{array}{ccc} \mathcal{S}(\lambda) - \mathrm{mod} - \mathcal{S}(n) & \xrightarrow{Q_\lambda \otimes_{\mathcal{S}(\lambda)} -} & C(\lambda) - \mathrm{mod} - \mathcal{S}(n) \\ {\scriptstyle \mathcal{S}(n)e_\lambda \otimes_{\mathcal{S}(\lambda)} -}\downarrow & & \downarrow{\scriptstyle R^G_{L_\lambda}} \\ \mathcal{S}(n) - \mathrm{mod} - \mathcal{S}(n) & \xrightarrow{Q \otimes_{\mathcal{S}(n)} -} & C(n) - \mathrm{mod} - \mathcal{S}(n) \end{array}$$

We will show that for q-Schur algebras multiplying with the idempotent e_λ corresponds to Harish-Chandra restriction. More precisely, we will show that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{S}(n) - \mathrm{mod} - \mathcal{S}(n) & \xrightarrow{Q \otimes_{\mathcal{S}(n)} -} & C(n) - \mathrm{mod} - \mathcal{S}(n) \\ {\scriptstyle e_\lambda .}\downarrow & & \downarrow{\scriptstyle {}^*R^G_{L_\lambda}} \\ \mathcal{S}(\lambda) - \mathrm{mod} - \mathcal{S}(n) & \xrightarrow{Q_\lambda \otimes_{\mathcal{S}(\lambda)} -} & C(\lambda) - \mathrm{mod} - \mathcal{S}(n). \end{array}$$

That is the following holds

Lemma 5.3. *For any $\mathcal{S}(n)$-$\mathcal{S}(n)$-bimodule N, there is an $C(\lambda)$-$\mathcal{S}(n)$-bimodule isomorphism*

$${}^*\mathrm{R}^G_{L_\lambda}\, Q \otimes_{\mathcal{S}(n)} N \cong Q_\lambda \otimes_{\mathcal{S}(\lambda)} e_\lambda N.$$

Proof. The Morita equivalence between the subalgebras $\mathcal{S}(\lambda)$ and $C(\lambda)$ induces an isomorphism of $C(\lambda)$-$\mathcal{S}(n)$-bimodules

$${}^*\mathrm{R}^G_{L_\lambda}\, Q \otimes_{\mathcal{S}(n)} N \cong Q_\lambda \otimes_{\mathcal{S}(\lambda)} \mathrm{Hom}_{C(\lambda)}(Q_\lambda, C_\lambda) \otimes_{C(\lambda)} {}^*\mathrm{R}^G_{L_\lambda} Q \otimes_{\mathcal{S}(n)} N.$$

As Q_λ is a projective $C(\lambda)$-module by [1, 20.10] the term on the right hand side is isomorphic to $Q_\lambda \otimes_{\mathcal{S}(\lambda)} \mathrm{Hom}_{C(\lambda)}(Q_\lambda, {}^*\mathrm{R}_{L_\lambda}^G\, Q \otimes_{\mathcal{S}(n)} N)$. By [4, 4.2] $R_{L_\lambda}^G Q_\lambda$ and Qe_λ are isomorphic as $C(n)$-$\mathcal{S}(\lambda)$-bimodules. Therefore adjunction of Harish-Chandra induction and restriction gives an isomorphism

$$Q_\lambda \otimes_{\mathcal{S}(\lambda)} \mathrm{Hom}_{C(\lambda)}(Q_\lambda, {}^*\mathrm{R}_{L_\lambda}^G\, Q \otimes_{\mathcal{S}(n)} N) \cong Q_\lambda \otimes_{\mathcal{S}(\lambda)} \mathrm{Hom}_{C(n)}(Qe_\lambda, Q \otimes_{\mathcal{S}(n)} N).$$

Now applying [1, 20.10] again, we obtain

$$\mathrm{Hom}_{C(n)}(Qe_\lambda, Q \otimes_{\mathcal{S}(n)} N) \cong \mathrm{Hom}_{C(n)}(Qe_\lambda, Q) \otimes_{\mathcal{S}(n)} N$$

where $\mathrm{Hom}_{C(n)}(Qe_\lambda, Q) \cong e_\lambda \mathrm{End}_{C(n)}(Q) \cong e_\lambda \mathcal{S}(n)$. The result follows. □

In particular, the two squares above still commute when we replace $C_{\mathcal{O}}(n)$ by $\mathcal{O}Gf$ and $C_{\mathcal{O}}(\lambda)$ by $\mathcal{O}L_\lambda f_\lambda$ as well as when the horizontal arrows are reversed, that is when $Q \otimes_{\mathcal{S}(n)} -$ is replaced by $Q' \otimes_{\mathcal{O}Gf} -$ and $Q_\lambda \otimes_{\mathcal{S}(\lambda)} -$ is replaced by $Q'_\lambda \otimes_{\mathcal{O}L_\lambda f_\lambda} -$.

In order to show that tensoring with $X_{\mathcal{S}}$ induces a derived equivalence, we first show that $X_{\mathcal{S}}$ is isomorphic to the following complex Y. Let $F = Q' \otimes_{\mathcal{O}Gf} - \otimes_{\mathcal{O}Gf} Q$ be the bimodule functor induced by the Morita equivalence between $\mathcal{O}Gf$ and $\mathcal{S}_{\mathcal{O}}(n)$ and set

$$Y = F(fX_G f).$$

Then Y is a complex of $\mathcal{S}_{\mathcal{O}}(n)$-$\mathcal{S}_{\mathcal{O}}(n)$-bimodules with degree ith term

$$Y^i = \bigoplus_{\lambda \in \bar{\Lambda}(n-i,n)} Q' e_{U_\lambda} \otimes_{\mathcal{O}L_\lambda f_\lambda} e_{U_\lambda} Q$$

for $0 \le i \le n-1$ and with differential

$$d^i = \sum_{\lambda \in \bar{\Lambda}(n-i,n)} \sum_{\substack{\nu \in \bar{\Lambda}(n-i-1,n) \\ \lambda \prec \nu}} (-1)^{\lambda/\nu} F(\alpha_{\lambda,\nu}).$$

Proposition 5.4. *The complexes $X_{\mathcal{S}}$ and Y are isomorphic as complexes of $\mathcal{S}_{\mathcal{O}}(n)$-$\mathcal{S}_{\mathcal{O}}(n)$-bimodules.*

Proof. We start by showing the terms in each degree are isomorphic bimodules. Let λ be a composition of n, then $\mathcal{S}(n)e_\lambda \otimes_{\mathcal{S}(\lambda)} e_\lambda \mathcal{S}(n) = \mathcal{S}(n)e_\lambda \otimes_{\mathcal{S}(\lambda)} Q'_\lambda \otimes_{\mathcal{S}(\lambda)} Q_\lambda \otimes_{\mathcal{S}(\lambda)} e_\lambda \mathcal{S}(n)$. By the preceding lemma $Q_\lambda \otimes_{\mathcal{S}(\lambda)} e_\lambda \mathcal{S}(n) \cong e_{U_\lambda} Q$. In a similar way $\mathcal{S}(n)e_\lambda \otimes_{\mathcal{S}(\lambda)} Q'_\lambda otimes_{\mathcal{S}(\lambda)} e_{U_\lambda} Q$ is isomprphic to $Q' \otimes_{\mathcal{O}Gf} \mathcal{O}Gf e_{U_\lambda} \otimes_{\mathcal{O}L_\lambda f_\lambda} e_{U_\lambda} Q \cong Q' e_{U_\lambda} \otimes_{\mathcal{O}L_\lambda f_\lambda} e_{U_\lambda} Q$.

Then as all isomorphisms involved are functorial, they clearly commute with the differentials. □

Proof of theorem 5.1. The functor

$$Q' \otimes_{\mathcal{O}Gf} - \otimes_{\mathcal{O}Gf} Q \ : \ \mathcal{O}Gf - \mathrm{mod} - \mathcal{O}Gf \ \longrightarrow \ \mathcal{S}(n) - \mathrm{mod} - \mathcal{S}(n)$$

is an equivalence of bimodule categories. The complex Y is isomorphic to $Q' \otimes_{\mathcal{O}Gf} fX_G f \otimes_{\mathcal{O}Gf} Q$ and by [24] the functor $fX_G f \otimes_{\mathcal{O}Gf} -$ induces a self-derived equivalence of the module category of $\mathcal{O}Gf$. Therefore the functor $Y \otimes_{\mathcal{S}_{\mathcal{O}}(n)} -$ induces a self-equivalence of $\mathcal{D}^\flat(\mathcal{S}_{\mathcal{O}}(n))$. By [22, 2.2] the complex $k \otimes_{\mathcal{O}} Y = Y_k$ induces a self-derived equivalence of the module category of $\mathcal{S}_k(n)$.

Denote by $D_{\mathcal{S}}$ the induced map in the Grothendieck group $K_0(\mathcal{S}_R(n))$ for $R \in \{K, k\}$. ia the Morita equivalence between $\mathcal{S}(n)$ and $C(n)$, he standard module $\Delta(\lambda)$ corresponds to the Specht module $S(1, \lambda')$ labelled by the conjugate partition and the costandard module $\nabla(\lambda)$ corresponds to the dual Specht module $S(1, \lambda)^*$ (see [25] or [4]). It follows then from proposition 3.1 and the above that $D_{\mathcal{S}}$ sends $\Delta(\lambda)$ to $\Delta(\lambda')$ and $\nabla(\lambda)$ to $\nabla(\lambda)'$ in $K_0(\mathcal{S}_R(n))$. Furthermore, $L(\lambda)$ is the head of $\Delta(\lambda)$ and $P\mathcal{Z}_u P^{-1}$ is the decomposition matrix for $\mathcal{S}_k(n)$. Thus part *(iii)* of the theorem follows.

□

6. Decomposition matrices

In this last section we recapitulate the information we have obtained on the different decomposition matrices. Recall the notation $\mathcal{Z}_G$ and $\mathcal{Z}_u$ for the decomposition matrix of G and its unipotent part. Set $\mathcal{Z}_u^e$ for the matrix given by the rows and columns of $\mathcal{Z}_u$ indexed by e-regular partitions of n. Denote by $\mathcal{Z}_{\mathcal{H}}$ the decomposition matrix of $\mathcal{H}$, that is the matrix whose coefficients are given by $[S^\lambda : D^\nu]$ for λ a partition of n and ν an e-regular partition of n. Finally denote by $\mathcal{Z}_{\mathcal{S}}$ the decomposition matrix of the q-Schur algebra $\mathcal{S}(n)$ with coefficients given by $[\Delta(\lambda) : L(\nu)]$ for partitions λ, ν of n. Recall that $\mathcal{Z}_{\mathcal{S}} = P\mathcal{Z}_u P^{-1}$.

Then combining the previous results we have the following relations

Theorem 6.1. *The following matrix equations hold*

(i) $A_G = \mathcal{Z}_u^{-1} P \mathcal{Z}_u$

(ii) $\mathcal{Z}_u^e A_{\mathcal{H}} = P \mathcal{Z}_u^e$

(iii) $A_{\mathcal{S}} = \mathcal{Z}_{\mathcal{S}}^{-1} P \mathcal{Z}_{\mathcal{S}} = P^{-1} A_G P.$

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Inst.f.Algebra u.Zahlentheorie, Universität Stuttgart, 70550 Stuttgart, Germany
E-mail address: ackermann@mathematik.uni-stuttgart.de

Mathematical Institute, 24-29 St Giles, Oxford OX1 3LB, United Kingdom
E-mail address: schroll@maths.ox.ac.uk